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Exact renormalization group: a new method for blocking the action

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Abstract
We consider the exact renormalization group for a non-canonical scalar field theory in which the field is coupled to the external source in a special nonlinear way. The Wilsonian action and the average effective action are then simply related by a Legendre transformation up to a trivial quadratic form. An exact mapping between canonical and non-canonical theories is obtained as well as the relations between their flows. An application to the theory of liquids is sketched.

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1. Introduction

During the last 20 years the Wilson approach [1] to the renormalization group (RG) has been the subject of a revival in both statistical and quantum field theory and also, quite independently, in the more restricted domain of the equilibrium statistical physics of classical liquids.

In field theory two main formulations of the non-perturbative renormalization group (NPRG) have been developed in parallel. In the first one, a continuous realization of the RG transformation of the action $\mathcal{S}_k[\varphi]$ is made and no expansion is involved with respect to some small parameter of this action. At scale-k (in momentum space) the high-energy modes $\widetilde{\varphi}_q, q > k$, have been integrated out in the 'Wilsonian' action $\mathcal{S}_k$ which is a functional of the slow modes $\widetilde{\varphi}_q, q < k$. This operation requires the implementation of some cut-off of the propagator aiming at separating slow ($q < k$) and fast ($q > k$) modes. The flow of the action is governed either by the Wilson–Polchinski equation [1–3] in the case of a smooth cut-off or the Wegner–Houghton [4] equation in the case of a sharp cut-off. These equations, due to their complexity, call for the use of approximation and/or truncation methods which have been extensively studied in the last few years; we refer to the review of Bagnuls and Bervillier [5] for a detailed discussion of this first version of the NPRG.

The second, more recent formulation, called the 'effective average action' approach, was developed after the seminal works of Nicoll, Chang and Stanley for the sharp cut-off version

[6, 7] and Wetterich, Ellwanger and Morris (WEM) for the smooth cut-off version [8–13]. This method implements on the effective average action $\Gamma_k[\Phi]$—roughly speaking the Gibbs free energy of the fast modes $\widetilde{\Phi}_q, q > k$ of the classical field—rather than on the Wilsonian action $\mathcal{S}_k$, the ideas of integration of high-energy modes that underlie any RG approach. The flow of Γ_k results in equations which can be solved under the same kind of non-perturbative approximations as those used for the Wilson–Polchinski or Wegner–Houghton equations. The main advantage of this more recent formulation is that it gives access to the RG flow of physical quantities, i.e. the Gibbs free energy $\Gamma_k[\Phi]$ and the correlation functions as well, rather than such a highly abstract object as the Wilsonian action. Recent reviews and lectures devoted to this second approach are available [14, 15] and should be consulted for a thorough discussion. These two versions of the RG are in fact equivalent; this not so obvious equivalence is discussed in detail by Morris in a beautiful paper [13].

As can be tracked back in the literature, the 'effective average action' approach of the RG was in fact discovered independently by Parola and Reatto in the framework of the theory of liquids nearly 25 years ago; they considered both the sharp and the soft cut-off formulation of the so-called hierarchical reference theory (HRT) [16–18]; a review article describes their early achievements [19] and several papers describing new developments of the soft cut-off version of HRT appeared recently [20, 21].

Some years ago it was realized that a statistical field description of liquids was possible and the so-called KSSHE theory of liquids (after the names of Kac, Siegert, Stratonovich, Hubbard and Edwards [22–26]) was introduced and developed in [27–32]. In [33] it was shown that the WEM equations for KSSHE field theory are identical to HRT equations in the sharp cut-off limit. There are however differences for the soft formulations and a picture of the RG of liquids in terms of a Wilsonian action does not emerge obviously from these early attempts.

A close inspection of KSSHE theory reveals that it is not an 'ordinary' or 'canonical' field theory in the sense that the coupling between the scalar 'internal' field and the 'external' source, in this particular case the chemical potential, is nonlinear. So it is slightly at variance with the usual formulations of field theory where a linear coupling is adopted in general. It turns out that the full RG construction is much easier for a KSSHE-like theory than for a canonical one. Therefore ideas pertaining to the theory of liquids can be exported to statistical field theory, yielding important simplifications for the latter. Indeed the subtle reasonings of Morris [13] can then be reproduced with a disarming simplicity by the introduction of a 'reference', non-Gaussian system; in this way we find that the Wilsonian action $\mathcal{S}_k$ and the WEM action Γ_k are related by a simple Legendre transformation (up to a trivial quadratic form). This is the main result of this paper.

The paper is organized as follows. In section 2 we show how to build a non-canonical, KSSHE-like field theory from a canonical one. We then follow Morris's construction of the RG in section 3, obtain the RG flows and discuss the interplay between the Wilson–Polchinsy and WEM formulations of the renormalization group. The exact mapping onto a related canonical theory is discussed in section 4. The full set of flow equations for the effective vertices is then discussed in section 5 for the soft and sharp cut-off versions. Finally, in section 6 we give an illustration for the theory of liquids and we conclude.

In order to simplify our discussions we have restricted ourselves to the case of bosonic scalar field theories; extensions to more complicated cases are certainly possible. Moreover we discuss here only the 'first step' of the RG program of Wilson, i.e. the blocking of the action and we make no comments or digressions on the scaling properties of the solutions of NPRG equations near a fixed point; specialists are still at variance on this point, see, e.g. [34].

2. A non-canonical statistical field theory

To simplify the discussion let us consider a 'reference' (R) system described by a standard scalar field theory. Other representations or generalizations are easy to deal with, as illustrated in section 6 where the case of the theory of liquids is briefly evoked. The physics of the R-system, i.e. its thermodynamics and correlation functions, is supposed to be known exactly; it is encoded in the functional [35, 36]

$$Z_R[J] = \int \mathcal{D}\chi \exp(-S_R[\chi] + J \cdot \chi), \tag{1}$$

where $J(x)$ is an external source and the action $S_R[\chi]$ is an arbitrary functional of the real scalar field χ; notably $S_R[\chi]$ might comprise terms linear or quadratic in χ. In (1) $J \cdot \chi$ is a short-cut for $\int_x J(x) \cdot \chi(x)$ where $\int_x \equiv \int \mathrm{d}^d x$ and d the space dimensions.

We denote by $W_R[J] = \ln Z_R[J]$ the Helmholtz free energy functional. As well known Z_R and W_R are the generators of ordinary and connected correlation functions which will be written as

$$Z_R^{(n)}(J; 1, 2, \ldots, n) = \frac{1}{Z_R} \frac{\delta^n Z_R}{\delta J(1) \cdots \delta J(n)}, \tag{2a}$$

$$W_R^{(n)}(J; 1, 2, \ldots, n) = \frac{\delta^n W_R}{\delta J(1) \cdots \delta J(n)}, \tag{2b}$$

where we used the uncluttered notations $i \equiv x_i$.

From first principles $W_R[J]$ is a convex functional of the source $J(x)$. Its Legendre–Fenchel transform $\Gamma_R[\Phi]$, the reference Gibbs free energy, is therefore also a convex functional of the classical field $\Phi(x)$. We thus have

$$\Gamma_R[\Phi] = \sup_J (J \cdot \Phi - W_R[J]), \tag{3a}$$

$$W_R[J] = \sup_\Phi (J \cdot \Phi - \Gamma_R[\Phi]), \tag{3b}$$

from which we deduce Young inequalities

$$\Gamma_R[\Phi] + W_R[J] \geqslant J \cdot \Phi \qquad (\forall \Phi, \forall J), \tag{4}$$

which may be used to obtain rigorous bounds (see, e.g., appendix B).

It will prove useful to introduce the proper vertex functions of the R-system

$$\Gamma_R^{(n)}(\Phi; 1, 2, \ldots, n) = \frac{\delta^n \Gamma_R}{\delta \Phi(1) \ldots \delta \Phi(n)}. \tag{5}$$

The Legendre–Fenchel transform (3) is more general than, but in the cases that will be considered here equivalent to, the usual Legendre transform defined as

$$\Gamma_R[\Phi] + W_R[J] = J \cdot \Phi \qquad \begin{cases} \forall \Phi & J(x) = \Gamma_R^{(1)}(\Phi; x), \\ \forall J & \Phi(x) = W_R^{(1)}(J; x). \end{cases} \tag{6}$$

Our requirements concerning the properties of the R-system will be modest and fuzzy; a reasonable assumption is that it is not at, or too close to a critical point, so that connected correlation functions are short ranged and Taylor functional expansions about some arbitrary field make sense. In practice $Z_R[J]$ is of course not known exactly and will in general result from some approximation, a high-temperature expansion for instance. The choice of a Gaussian model for the R-system would obviously be either of little interest or a lack of ambition. At this point we introduce and want to study a family of models (referred to as Λ-systems) labelled by Λ in momentum space and built as follows:

$$Z_{\Lambda}[J] = \int \mathcal{D}\chi \exp\left(-S_R[\chi] + \frac{1}{2}\chi \cdot P_0^{\Lambda} \cdot \chi + J \cdot \chi\right), \tag{7}$$

where $\chi \cdot P_0^{\Lambda} \cdot \chi \equiv \int_x \int_y \chi(x) P_0^{\Lambda}(x-y)\chi(y)$, which can also be rewritten as $\int_q \widetilde{P}_0^{\Lambda}(q)\chi_q\chi_{-q}$ in Fourier space where $\widetilde{P}_0^{\Lambda}$ and χ_q denote the Fourier transforms of $P_0^{\Lambda}(x)$ and $\chi(x)$ respectively, finally $\int_q \equiv \int \mathrm{d}^d q/(2\pi)^d$. We assume that P_0^{Λ} is definite positive (i.e. $\widetilde{P}_0^{\Lambda}(q) > 0$). Λ acts as an ultraviolet (UV) cut-off since

$$\widetilde{P}_0^{\Lambda} = \widetilde{P}_0(q) C\left(\frac{q}{\Lambda}\right), \tag{8a}$$

$$C(x) = 1 - \Theta_{\epsilon}(x - 1), \tag{8b}$$

where $\Theta_{\epsilon}(x)$ is a smoothened version of the step function $\Theta(x)$, ϵ being the range of the interval $(-\epsilon/2, \epsilon/2)$ where $\Theta_{\epsilon}(x)$ increases gently from $\Theta_{\epsilon} = 0$ to $\Theta_{\epsilon} = 1$. We will denote similarly by $\delta_{\epsilon}(x) = \partial\Theta_{\epsilon}(x)/\partial x$ the smoothened version of Dirac distribution. Taking (carefully) the limit $\epsilon \to 0$ yields the sharp cut-off version of the theory. In (8) $\widetilde{P}_0(q) \propto 1/(q^2 + m^2)$ is a massive propagator, but we can find no reason why m^2 could not be set to 0 if necessary. We see that $\widetilde{P}_0^{\Lambda}(q) \approx \widetilde{P}_0(q)$ for $q \leqslant \Lambda - \epsilon$ and $\widetilde{P}_0^{\Lambda}(q) \approx 0$ for $q \geqslant \Lambda + \epsilon$. The UV cut-off Λ may be understood as the scale at which the Λ-system is defined at a microscopic level; for an Ising model typically $\Lambda \approx 1/a$ where a is the lattice spacing and for a fluid of molecules of size σ, $\Lambda \approx 1/\sigma$. Note that since a positive quadratic term has been *subtracted* from the action S_R the Λ-system can be tuned to a critical point.

We now take advantage of the positivity of operator P_0^{Λ} to perform a Hubbard–Stratonovich transform [22–26] in (7) which yields

$$Z_{\Lambda}[J] = \frac{1}{\mathcal{N}_{P_0^{\Lambda}}} \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi \cdot R_0^{\Lambda} \cdot \varphi + W_R[J + \varphi]\right), \tag{9a}$$

$$\mathcal{N}_{P_0^{\Lambda}} = \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi \cdot R_0^{\Lambda} \cdot \varphi\right), \tag{9b}$$

where $R_0^{\Lambda} \equiv \left[P_0^{\Lambda}\right]^{-1}$ is the inverse of P_0^{Λ} in the sense of operators, i.e.

$$\int_y R_0^{\Lambda}(x, y) P_0^{\Lambda}(y, z) = \delta^d(x - z).$$

The Hubbard–Stratonovich transform and other useful properties of Gaussian functional integrals are reviewed in appendix A.

The field theory given by (9) is non-canonical in the sense that the coupling between the external source J and the field φ is a nonlinear one. This kind of field theory appears naturally in the statistical mechanics of simple fluids, the Ising model, etc after performing a Hubbard–Stratonovich transform in order to introduce a field theory for the model under consideration. The KSSHE theory of liquids is introduced and discussed in [27–32], some of its salient features are reviewed in appendix B and additional comments are given in section 6.

3. The exact renormalization group

3.1. Blocking the action

We now apply the exact RG approach of Tim Morris [13] to our non-canonical field theory. As a consequence of the Bogolioubov theorem (cf equation (A.7) in appendix A) the partition function $Z_\Lambda[J]$ can be rewritten in terms of two propagators and two fields as

$$Z_\Lambda[J] = \frac{1}{\mathcal{N}_{P_0^k}} \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi_< \cdot R_0^k \cdot \varphi_<\right) Z_k^\Lambda[\varphi_<, J], \tag{10a}$$

$$Z_k^\Lambda[\varphi_<, J] = \frac{1}{\mathcal{N}_{P_k^\Lambda}} \int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi_> \cdot R_k^\Lambda \cdot \varphi_> + W_R\left[J + \varphi_< + \varphi_>\right]\right), \tag{10b}$$

where $0 \leqslant k \leqslant \Lambda$ is the running scale of the RG and where

$$\varphi = \varphi_< + \varphi_> \qquad \text{and} \qquad P_0^\Lambda = P_k^\Lambda + P_0^k. \tag{11}$$

In (10)–(11) we have separated the field φ into 'rapid' ($\varphi_>$) and slow modes ($\varphi_<$). The low-energy modes are associated with the propagator P_0^k (with inverse R_0^k) which is cut-off from above by k, while the high-energy modes are associated with the propagator P_k^Λ (with inverse R_k^Λ) which is cut-off from below by k and from above by Λ. We demand that $\widetilde{P}_k^\Lambda(q) = \widetilde{P}_0(q)(C(q/\Lambda) - C(q/k))$ should be positive and thus the cut-off function $C(x)$ must be a decreasing function of its argument which will be assumed henceforth.

As in the canonical case, the functional $Z_k^\Lambda\left[\varphi_<, J\right]$ is the crux of the whole matter since it allows us to make explicit the link between the Wilsonian action and the effective average action [13]. However here this link proves trivial since $Z_k^\Lambda\left[\varphi_<, J\right]$ is a functional of the single variable $\varphi_< + J$.

Let us first set $J = 0$ in (10). On the one hand we have

$$Z_k^\Lambda[\varphi_<, J = 0] \triangleq \exp\left(-S_k^\Lambda[\varphi_<]\right) \tag{12a}$$

$$= \frac{1}{\mathcal{N}_{P_k^\Lambda}} \int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi_> \cdot R_k^\Lambda \cdot \varphi_> + W_R\left[\varphi_< + \varphi_>\right]\right), \tag{12b}$$

and on the other hand

$$Z_\Lambda[J = 0] = \frac{1}{\mathcal{N}_{P_0^k}} \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi_< \cdot R_0^k \cdot \varphi_< - S_k^\Lambda\left[\varphi_<\right]\right). \tag{12c}$$

Equations (12) define the Wilsonian action $S_k^\Lambda\left[\varphi_<\right]$ in the usual way, i.e. as the effective action of the slow modes at scale k [1, 2, 13]. Here k plays the role of an UV cut-off.

Let us now set $\varphi_< = 0$ in (10). It yields

$$Z_k^\Lambda\left[\varphi_< = 0, J\right] \triangleq Z_k^\Lambda\left[J\right] \left(\triangleq \exp\left(W_k^\Lambda\left[J\right]\right)\right) \tag{13a}$$

$$= \frac{1}{\mathcal{N}_{P_k^\Lambda}} \int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi_> \cdot R_k^\Lambda \cdot \varphi_> + W_R\left[J + \varphi_<\right]\right), \tag{13b}$$

which shows that $W_k^\Lambda\left[J\right]$ is the Helmholtz free energy of the rapid modes $\varphi_>$ in the presence of the source J; therefore, here, k plays the role of an infra-red (IR) cut-off. We will see in section 4 how $W_k^\Lambda\left[J\right]$ may also be seen, in some sense, as the generator of connected correlation functions with UV regularization, i.e. Λ, and IR cut-off, i.e. k.

We also note that the partition function $Z_k^\Lambda[J]$ can alternatively be written as a functional integral over the field χ, i.e.

$$Z_k^\Lambda[J] = \int \mathcal{D}\chi \exp\left(-S_R[\chi] + \frac{1}{2}\chi \cdot P_k^\Lambda \cdot \chi + J \cdot \chi\right). \tag{13c}$$

A Hubbard–Stratonovich transform allows us, indeed, to obtain (13*b*) from (13*c*) in the same way used to obtain (9) from the expression (7) of Z_Λ.

As a trivial consequence of (13*c*) we note that $W_\Lambda^\Lambda[J] = W_R[J]$ (since $P_\Lambda^\Lambda \equiv 0$ as follows from (11)) and $W_0^\Lambda[J] = W_\Lambda[J]$. Another important consequence of (13*c*) is the convexity of the functional $W_k^\Lambda[J]$ which follows from the usual arguments [36, 38].

The two approaches of the RG, that of the Wilsonian action and that of the effective average action, are here trivially related since

$$W_k^\Lambda = -S_k^\Lambda, \tag{14}$$

from which we infer that

$$Z_k^\Lambda[\varphi_<, J] = \exp\left(-S_k^\Lambda[\varphi_< + J]\right) = \exp\left(W_k^\Lambda[\varphi_< + J]\right). \tag{15}$$

When (15) is reported in (10) we get the illuminating expression

$$Z_\Lambda[J] = \frac{1}{\mathcal{N}_{P_0^k}} \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi \cdot R_0^k \cdot \varphi + W_k^\Lambda[J + \varphi]\right), \tag{16}$$

which, when compared to equation (9), shows that W_k^Λ can also be interpreted as a reference Helmholtz free energy at scale k or as the Helmholtz free energy of the k-system to paraphrase Parola and Reatto [18, 19].

3.2. Flow equations

3.2.1. The Helmholtz free energy W_k^Λ. We first establish the flow equation for W_k^Λ. It follows from expression (13*b*) and the algebraic identity (A.8*a*) of appendix A that

$$\exp\left(W_k^\Lambda[J]\right) = \exp\left(D_k^\Lambda\right)\exp(W_R[J]), \tag{17}$$

where

$$D_k^\Lambda \equiv \frac{1}{2}\int_{x,y} P_k^\Lambda(x, y)\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}. \tag{18}$$

Taking partial derivatives of both sides of equation (17) with respect to k at fixed $J(x)$ yields

$$\partial_k W_k^\Lambda[J]|_J = \frac{1}{2}\int_{x,y} \partial_k P_k^\Lambda(x, y)\left\{W_k^{\Lambda(2)}(x, y) + W_k^{\Lambda(1)}(x)W_k^{\Lambda(1)}(y)\right\}. \tag{19}$$

This flow equation must be supplemented by the initial condition $W_\Lambda^\Lambda = W_R$ at $k = \Lambda$.

3.2.2. The Wilsonian action S_k^Λ. Since $W_k^\Lambda = -S_k^\Lambda$ the flow of S_k^Λ is given by the usual Wilson–Polchinski equation [3, 13]

$$\partial_k S_k^\Lambda[\Phi]|_\Phi = \frac{1}{2}\int_{x,y} \partial_k P_k^\Lambda(x, y)\left\{S_k^{\Lambda(2)}(x, y) - S_k^{\Lambda(1)}(x)S_k^{\Lambda(1)}(y)\right\}, \tag{20}$$

to be supplemented with the initial condition $S_\Lambda^\Lambda = -W_R$ at $k = \Lambda$.

3.2.3. The effective average action Γ_k^Λ. The 'true' Gibbs free energy of the k-system, provisionally denoted as $\overline{\Gamma}_k^\Lambda$, is defined as the Legendre transformation of $W_k^\Lambda[J]$ and we thus have the couple of relations

$$\overline{\Gamma}_k^\Lambda[\Phi] = \sup_J \left(J \cdot \Phi - W_k^\Lambda[J] \right), \tag{21a}$$

$$W_k^\Lambda[J] = \sup_\Phi \left(J \cdot \Phi - \overline{\Gamma}_k^\Lambda[\Phi] \right), \tag{21b}$$

where both $W_k^\Lambda[\Phi]$ and $\overline{\Gamma}_k^\Lambda[\Phi]$ are convex functionals of their arguments. Then it follows from stationarity that $\partial_k \overline{\Gamma}_k^\Lambda[\Phi]\big|_\Phi = -\partial_k W_k^\Lambda[J]\big|_J$ ($\forall J$, provided that $\Phi(x) = \delta W_k^\Lambda \big/ \delta J(x)$ or, $\forall \Phi$, provided that $J(x) = \delta \overline{\Gamma}_k^\Lambda / \delta\Phi(x)$ [35]) from which we conclude that

$$\partial_k \overline{\Gamma}_k^\Lambda[\Phi]|_\Phi = \frac{1}{2} \int_{x,y} \partial_k P_0^k(x, y) \left\{ W_k^{\Lambda(2)}(x, y) + \Phi(x)\Phi(y) \right\}, \tag{22}$$

where we have used the fact that $\partial_k P_0^k = -\partial_k P_k^\Lambda$; we would like to point out that $W_k^{\Lambda(2)}$ is the inverse of $\overline{\Gamma}_k^{\Lambda(2)} = \Gamma_k^{\Lambda(2)} + P_0^k$ so that (22) is closed. To get rid of the non-local term on the right-hand side of the equation we are led to *define* the effective average action as

$$\Gamma_k^\Lambda[\Phi] = \overline{\Gamma}_k^\Lambda[\Phi] - \tfrac{1}{2}\Phi \cdot P_0^k \cdot \Phi. \tag{23}$$

Note that $\Gamma_k^\Lambda[\Phi]$ can be non-convex as long as $k > 0$ since the operator P_0^k is definite positive. Obviously its flow equation takes the simple form

$$\partial_k \Gamma_k^\Lambda[\Phi]|_\Phi = \frac{1}{2} \int_{x,y} \partial_k P_0^k(x, y) \left\{ \Gamma_k^{\Lambda(2)} + P_0^k \right\}^{-1}(x, y), \tag{24}$$

which coincides with WEM equation. This equation must be supplemented with an initial condition. From $W_\Lambda^\Lambda = W_R$ it follows that $\overline{\Gamma}_\Lambda^\Lambda = \Gamma_R$ and thus, from (23) we get

$$\Gamma_\Lambda^\Lambda[\Phi] = \Gamma_R[\Phi] - \tfrac{1}{2}\Phi \cdot P_0^\Lambda \cdot \Phi. \tag{25}$$

At this point some comments are in order. First, it turns out that, as shown in appendix B, expression (25) of $\Gamma_\Lambda^\Lambda[\Phi]$ coincides with the mean field (MF), or tree level approximation for the Gibbs potential $\Gamma_\Lambda[\Phi]$, which we denote by $\Gamma_{\mathrm{MF}}^\Lambda[\Phi]$. Therefore, as in the usual canonical case, the RG flow drives the effective average action $\Gamma_k^\Lambda[\Phi]$ from its MF value at $k = \Lambda$ to its exact value at $k = 0$ by integrating fluctuations of smaller and smaller wave numbers. Moreover it is also shown in appendix B that

$$\Gamma_k^\Lambda[\Phi] \leqslant \Gamma_{\mathrm{MF}}^\Lambda[\Phi] \qquad \forall \Phi(x), \tag{26}$$

i.e. $\Gamma_{\mathrm{MF}}^\Lambda[\Phi]$ constitutes an exact upper bound for the effective average action.

Second comment: the arguments which led us to obtain equation (17) can also well be applied to equations (9) and (16) which gives

$$\exp(W_\Lambda[J]) = \exp(D_0^k)\exp(W_k^\Lambda[J]), \tag{27a}$$

$$\exp\left(W_k^\Lambda[J]\right) = \exp(D_k^\Lambda)\exp(W_R[J]), \tag{27b}$$

$$\exp(W_\Lambda[J]) = \exp(D_0^\Lambda)\exp(W_R[J]), \tag{27c}$$

i.e. the nice semi-group law $e^{D_0^\Lambda} \ldots = e^{D_0^k} e^{D_k^\Lambda} \ldots$, which of course does not trivially follow from $D_0^\Lambda = D_0^k + D_k^\Lambda$ but in addition requires the 'time ordering' of operators $e^{D_0^k}$ and $e^{D_k^\Lambda}$ as well as the conditions $0 \leqslant k \leqslant \Lambda$.

3.3. Reparametrization invariance

We discuss briefly the reparametrization invariance of the theory; indeed, changing the UV cut-off from Λ to some $\Lambda' \leqslant \Lambda$ should not change the physics at scale k provided the 'new' reference system is properly reparametrized at scale Λ'. We will do it for S and Γ, the two faces of our Janus RG.

Let us choose some running wave number $0 \leqslant k \leqslant \Lambda' \leqslant \Lambda$. Recall that we have $e^{-S_k^\Lambda} = e^{D_k^\Lambda} e^{-S_R}$ with $S_R = -W_R$ and we define $S'_R = S^\Lambda_{\Lambda'}$. Obviously the semi-group law $e^{D_0^\Lambda} \ldots = e^{D_0^k} e^{D_k^\Lambda} \ldots$ which was proved to be valid for $0 \leqslant k \leqslant \Lambda$ in section 3.2.3 can be generalized without problems to the triplet $k \leqslant \Lambda' \leqslant \Lambda$ (the fact that the smallest wavenumber $k = 0$ in equations (27) plays no role) and therefore we have $e^{-S_k^\Lambda} = e^{D_k^{\Lambda'}} e^{D_{\Lambda'}^\Lambda} e^{-S_R} = e^{-S_k^{\Lambda'}}$ with $e^{-S_k^{\Lambda'}} = e^{D_k^{\Lambda'}} e^{-S'_R}$. Therefore $S_k^{\Lambda'} = S_k^\Lambda$ if the action of the new reference system is indeed chosen to be $S'_R = S^\Lambda_{\Lambda'}$; this proves the reparametrization invariance for the Wilsonian action S_k^Λ.

We turn now our attention to the effective average action Γ_k^Λ and give two derivations of the reparametrization invariance as both are instructive. Since $S = -W$ we have of course $W_k^{\Lambda'} = W_k^\Lambda$ and more generally $W_k^{\Lambda'(n)} = W_k^{\Lambda(n)}$. In particular the full propagators ($n = 2$) are equal and we infer from the form of the flow equation (24) that $\partial_k \Gamma_k^{\Lambda'}[\Phi] = \partial_k \Gamma_k^\Lambda[\Phi]$. It remains to examine the initial condition for $\Gamma_k^{\Lambda'}[\Phi]$ at $k = \Lambda'$. $\Gamma_{\Lambda'}^{\Lambda'}[\Phi]$ is given by equation (25) i.e.

$$\Gamma_{\Lambda'}^{\Lambda'}[\Phi] = \Gamma'_R[\Phi] - \tfrac{1}{2}\Phi \cdot P_0^{\Lambda'} \cdot \Phi.$$

Since $\Gamma'_R[\Phi] = \overline{\Gamma}_{\Lambda'}^\Lambda[\Phi]$ (a direct consequence of $S'_R = S^\Lambda_{\Lambda'} = -W^\Lambda_{\Lambda'}$) it follows from the very definition (23) that $\Gamma_{\Lambda'}^{\Lambda'}[\Phi] = \Gamma_{\Lambda'}^\Lambda[\Phi]$. Integrating the flow equations thus yields $\Gamma_k^{\Lambda'}[\Phi] = \Gamma_k^\Lambda[\Phi]$, i.e. the reparametrization invariance for the effective average action.

For a simpler proof we start from the reparametrization invariance for the Wilsonian action. As $S = -W$ we have $W_k^{\Lambda'}[J] = W_k^\Lambda[J]$ from which $\overline{\Gamma}_k^{\Lambda'}[\Phi] = \overline{\Gamma}_k^\Lambda[\Phi]$ by Legendre transform. Then it follows from equation (23) that $\Gamma_k^{\Lambda'}[\Phi] = \Gamma_k^\Lambda[\Phi]$; it was therefore of the utmost importance that the quadratic form subtracted from $\overline{\Gamma}_k^\Lambda[\Phi]$ to define $\Gamma_k^\Lambda[\Phi]$ did not depend explicitly on the UV cut-off Λ.

4. Mapping on the canonical theory

4.1. The mapping

Commenting on equation (9) we already stressed the non-canonical functional dependence of $Z_\Lambda[J]$ upon the source $J(x)$. This remark holds at any scale $0 \leqslant k \leqslant \Lambda$ and also applies to the partition function $Z_k^\Lambda[J]$ of the k-system. The k-independent change of variable

$$\varphi \to \varphi^\star = \varphi + J \tag{28}$$

in equation (13) obviously allows a simple mapping on a canonical theory. We shall distinguish by a superscript '*' all the quantities pertaining to this canonical theory. Substituting φ for $\varphi^\star$ in expression (13) of $Z_k^\Lambda[J]$ readily yields

$$Z_k^\Lambda[J] = \mathrm{e}^{-\frac{1}{2} J \cdot R_k^\Lambda \cdot J} Z_k^{\Lambda *}[J^*], \tag{29a}$$

where $Z_k^{\Lambda *}[J^*]$ reads as

$$Z_k^{\Lambda *}[J^*] = \frac{1}{\mathcal{N}_{P_k^\Lambda}} \int \mathcal{D}\varphi^* \exp\left(-\frac{1}{2}\varphi^* \cdot R_k^\Lambda \cdot \varphi^* + W_R[\varphi^*] + J^* \cdot \varphi^*\right), \tag{29b}$$

and where the sources J and J^* are related by the simple linear relations

$$J^* = R_k^\Lambda \cdot J \Leftrightarrow J = P_k^\Lambda \cdot J^*. \tag{29c}$$

$Z_k^{\Lambda *}[J^*]$ is the standard or 'canonical' form of the partition function of the k-system. One defines as usual the Helmholtz free energy as $W_k^{\Lambda *} = \ln Z_k^{\Lambda *}$. The construction of the Wilsonian action $S_k^{\Lambda *}$ is worked out by means of a Bogolioubov transformation as in section 3.1 and $\Gamma_k^{\Lambda *}$ is obtained by a modified Legendre transform of $W_k^{\Lambda *}$. Details of the derivations are to be found in the paper of Morris (cf [13]). We reproduce here only his key results, rewritten however within our notations. First, one has (compare with equations (10))

$$Z_\Lambda^*[J^*] = \frac{1}{\mathcal{N}_{P_0^k}} \int \mathcal{D}\varphi_<^* \exp\left(-\frac{1}{2}\varphi_<^* \cdot R_0^k \cdot \varphi_<^*\right) Z_k^{\Lambda *}[\varphi_<^*, J^*], \tag{30a}$$

$$Z_k^{\Lambda *}[\varphi_<^*, J^*] = \frac{1}{\mathcal{N}_{P_k^\Lambda}} \int \mathcal{D}\varphi_>^* \exp\left(-\frac{1}{2}\varphi_>^* \cdot R_k^\Lambda \cdot \varphi_>^* + W_R[\varphi_<^* + \varphi_>^*] \cdots + J^* \cdot (\varphi_<^* + \varphi_>^*)\right). \tag{30b}$$

As in section 3.1 the two-fields functional $Z_k^{\Lambda *}[\varphi_<^*, J^*]$ is the key to the Janus temple. Making $J = 0$ in (30) defines the Wilsonian action

$$\begin{aligned} Z_k^{\Lambda *}[\varphi_<^*, J^* = 0] &= \exp\left(-S_k^{\Lambda *}[\varphi_<^*]\right), \\ &= \exp\left(D_k^\Lambda\right) \exp\left(W_R[\varphi_<^*]\right), \end{aligned} \tag{31}$$

while letting $\varphi_<^* = 0$ defines the Helmholtz free energy

$$\begin{aligned} Z_k^{\Lambda *}[\varphi_<^* = 0, J^*] &= \exp\left(W_k^{\Lambda *}[J^*]\right), \\ &= \exp\left(\tfrac{1}{2} J^* \cdot P_k^\Lambda \cdot J^*\right) \exp\left(D_k^\Lambda\right) \exp(W_R[J^*]). \end{aligned} \tag{32}$$

Below we make explicit the mapping between all star and non-star quantities and compare their RG flows.

4.2. $W_k^\Lambda[J]$ *and the Green functions*

The canonical and non-canonical Helmholtz free energy therefore differ by a simple quadratic form and we have, for example, for $W_k^{\Lambda *}$ in terms of W_k^Λ

$$W_k^{\Lambda *}[J^*] = W_k^\Lambda[J] + \tfrac{1}{2} J \cdot R_k^\Lambda \cdot J, \tag{33}$$

as follows either from equations (29a) or (32) which therefore are thus indeed equivalent. $W_k^{\Lambda *}[J^*]$ is the generator of the connected correlation functions of field φ^*. Since φ and φ^* differ by a constant (cf (28)) their connected correlations differ only at order $n = 1$ for which $\langle \varphi^* \rangle = \langle \varphi \rangle + J$. We shall denote $\Phi^* \equiv W_k^{\Lambda *(1)} = \langle \varphi^* \rangle$ the order parameter and will adopt the same notation for its non-star counterpart $\Phi \equiv W_k^{\Lambda (1)}$ although it could be misleading since, in the non-canonical case, $\Phi \neq \langle \varphi \rangle$. Taking the functional derivative of both sides of (33) and making use of the linear relations (29c) between the sources J and J^* leads to the relations

$$\Phi = -J^* + R_k^\Lambda \cdot \Phi^*, \qquad \Phi^* = J + P_k^\Lambda \cdot \Phi. \tag{34}$$

By performing successive derivatives of the above relations with respect either to J or to J^* one obtains easily the searched for relation between the two sets of Green functions

$$\begin{aligned} &W_k^{\Lambda *(2)}(1,2) = P_k^\Lambda(1,2) + P_k^\Lambda(1,1')P_k^\Lambda(2,2')W_k^{\Lambda (2)}(1',2') \\ &W_k^{\Lambda *(n)}(1,\ldots,n) = P_k^\Lambda(1,1')\cdots P_k^\Lambda(n,n')W_k^{\Lambda (n)}(1',\ldots,n') \qquad \text{for} \quad n \geqslant 3, \end{aligned} \tag{35}$$

where summation, i.e. space integration, over repeated indices ($n \equiv x_n$) is meant (to unclutter notations, the functional dependence of Green functions upon the sources J and J^* was not displayed explicitly).

4.3. The Wilsonian action S_k^{Λ}

This one is easy; a serene contemplation of equations (31) and (12) should convince the reader that

$$S_k^{\Lambda *}[\Psi] = S_k^{\Lambda}[\Psi] \quad (\forall \Psi). \tag{36}$$

4.4. The effective average action Γ_k^{Λ}

Recall that, in the canonical case, as we did in section 3, one first introduces the Legendre transform $\overline{\Gamma}_k^{\Lambda}$ of the Helmholtz free energy

$$\overline{\Gamma}_k^{\Lambda *}[\Phi^*] + W_k^{\Lambda *}[J^*] = J^* \cdot \Phi^* \qquad \begin{cases} \forall \Phi^* & J^* = \delta \overline{\Gamma}_k^{\Lambda *} / \delta \Phi^*, \\ \forall J^* & \Phi^* = \delta W_k^{\Lambda *} / \delta J^*. \end{cases} \tag{37}$$

Recall that $W_k^{\Lambda *}[J^*]$ and $\overline{\Gamma}_k^{\Lambda *}[\Phi^*]$ are both convex functionals and that the (possibly non-convex) effective average action $\Gamma_k^{\Lambda *}[\Phi^*]$ is defined as [10, 13, 14]

$$\Gamma_k^{\Lambda *}[\Phi^*] = \overline{\Gamma}_k^{\Lambda *}[\Phi^*] - \tfrac{1}{2} \Phi^* \cdot R_k^{\Lambda} \cdot \Phi^*. \tag{38}$$

The mapping between the non-canonical and canonical average effective actions is obtained from the mapping (33) between the Helmholtz free energies. A straightforward calculation yields

$$\Gamma_k^{\Lambda *}[\Phi^*] = \Gamma_k^{\Lambda}[\Phi] - \frac{1}{2} \Phi \cdot P_0^{\Lambda} \cdot \Phi - \Phi \cdot \frac{\delta \Gamma_k^{\Lambda}}{\delta \Phi}, \tag{39a}$$

$$\Phi^* = P_0^{\Lambda} \cdot \Phi + \frac{\delta \Gamma_k^{\Lambda}}{\delta \Phi}, \tag{39b}$$

or equivalently, from the ‘star world’ to the ‘non-star world’

$$\Gamma_k^{\Lambda}[\Phi] = \Gamma_k^{\Lambda *}[\Phi^*] - \frac{\delta \Gamma_k^{\Lambda *}}{\delta \Phi^*} \cdot \Phi^* - \frac{1}{2} \frac{\delta \Gamma_k^{\Lambda *}}{\delta \Phi^*} \cdot P_0^{\Lambda} \cdot \frac{\delta \Gamma_k^{\Lambda *}}{\delta \Phi^*}, \tag{40a}$$

$$\Phi = - \frac{\delta \Gamma_k^{\Lambda *}}{\delta \Phi^*}. \tag{40b}$$

These expressions are quite complicated and, despite some efforts, we were unable to derive from them the mapping between the vertices $\Gamma_k^{\Lambda(n)}$ and $\Gamma_k^{\Lambda *(n)}$ for a general ‘n’ (quite nice-looking relations are easily obtained for $n \leqslant 3$ but cannot be generalized in a straightforward manner for higher ‘n’).

An instructive consequence of equations (39) and (40) is the derivation of the initial condition for $\Gamma_k^{\Lambda *}[\Phi^*]$. From expression (25) of $\Gamma_{\Lambda}^{\Lambda}[\Phi]$ combined with equation (39b) one gets $\Phi^* = \delta \Gamma_R / \delta \Phi$ ($\equiv J_R$ if you wish).

From (39a) one then infers

$$\begin{aligned} \Gamma_{\Lambda}^{\Lambda *}[\Phi^*] &= \Gamma_R[\Phi] - J_R \cdot \Phi, \\ &= -W_R[J_R], \\ &= S_{\Lambda}[\Phi^*], \end{aligned} \tag{41}$$

which is indeed the expected result [10, 13, 14].

Our final task is to relate the flows of the effective average actions in the canonical and non-canonical theories. The result is quite remarkable and reads as

$$\partial_k \Gamma_k^{\Lambda *}[\Phi^*]|_{\Phi^*} = \partial_k \Gamma_k^{\Lambda}[\Phi]|_{\Phi},$$
$$\text{with} \qquad \Phi = -\frac{\delta \Gamma_k^{\Lambda *}}{\delta \Phi^*} \qquad \text{or} \qquad \Phi^* = P_0^{\Lambda} \cdot \Phi + \frac{\delta \Gamma_k^{\Lambda}}{\delta \Phi}. \tag{42}$$

There are several proofs of this result, one of them being to start from equation (39). Taking its partial derivative with respect to 'k' at fixed Φ^* yields

$$\begin{aligned}\partial_k \Gamma_k^{\Lambda *}[\Phi^*]|_{\Phi^*} &= \partial_k \Gamma_k^{\Lambda}[\Phi]|_{\Phi^*} + \Phi \cdot P_0^{\Lambda} \cdot \partial_k \Phi|_{\Phi^*} - \Phi^* \cdot \partial_k \Phi|_{\Phi^*}\,,\\ &= \partial_k \Gamma_k^{\Lambda}[\Phi]|_{\Phi} + \partial_k \Phi|_{\Phi^*} \cdot \left\{ \frac{\delta \Gamma_k^{\Lambda}}{\delta \Phi} + P_0^{\Lambda} \cdot \Phi - \Phi^* \right\},\\ &= \partial_k \Gamma_k^{\Lambda}[\Phi]|_{\Phi} \text{QED},\end{aligned} \tag{43}$$

where we made use of (39b) to obtain the last line.

A second, more direct proof of equation (42) gives us the opportunity to write the well-known WEM equation for $\Gamma_k^{\Lambda *}$ which we present with simplified notations as

$$\partial_k \Gamma_k^{\Lambda *}[\Phi^*]|_{\Phi^*} = \tfrac{1}{2} \partial_k R_k^{\Lambda}(1,2) W_k^{\Lambda * (2)}(1,2) + \partial_k \ln \mathcal{N}_{P_k^{\Lambda}}. \tag{44}$$

The second contribution to the rhs of (44) involves the normalization $\mathcal{N}_{P_k^{\Lambda}}$; it is independent of the field and for that reason generally not mentioned in the literature; here, however, we need it to complete our proof. Clearly

$$\begin{aligned}\partial_k \ln \mathcal{N}_{P_k^{\Lambda}} &= -\tfrac{1}{2} \langle \varphi(1)\varphi(2) \rangle_{P_k^{\Lambda}} \, \partial_k R_k^{\Lambda}(1,2)\\ &= -\tfrac{1}{2} P_k^{\Lambda}(1,2) \partial_k R_k^{\Lambda}(1,2),\end{aligned} \tag{45}$$

where the brackets denote a Gaussian average (see appendix A) and we made use of Wick's theorem. To go further we remark that $\partial_k R_k^{\Lambda} = -R_k^{\Lambda} \cdot \partial_k P_k^{\Lambda} \cdot R_k^{\Lambda}$ and also make use of the relations between canonical and non-canonical Green functions (cf equations (35)). This gives us

$$\begin{aligned}\partial_k \Gamma_k^{\Lambda *}[\Phi^*]|_{\Phi^*} &= -\tfrac{1}{2} \partial_k P_k^{\Lambda}(1,2) \left\{ W_k^{\Lambda (2)}(1,2) + R_k^{\Lambda}(1,2) \right\}\\ &\quad \ldots - \tfrac{1}{2} P_k^{\Lambda}(1,2) \partial_k R_k^{\Lambda}(1,2)\\ &= -\tfrac{1}{2} \partial_k P_k^{\Lambda}(1,2) W_k^{\Lambda (2)}(1,2)\\ &= \tfrac{1}{2} \partial_k P_0^{k}(1,2) W_k^{\Lambda (2)}(1,2),\end{aligned} \tag{46}$$

which indeed is equal to $\partial_k \Gamma_k^{\Lambda}[\Phi]|_{\Phi}$ (cf equation (24)).

5. The RG hierarchy

5.1. Smooth cut-off

In this section we comment on the flow equation (24) for the average effective action of our non-canonical KSSHE-like field theory. Henceforth we consider only homogeneous systems, then, as usual, in momentum space, we factor out and evaluate the momentum conserving δ function so n-point correlation functions $\widetilde{\Gamma}_k^{\Lambda (n)}(q_1, \ldots, q_n)$ are defined only when $q_1 + \cdots + q_n = 0$. More precisely one has for instance

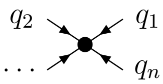
$$\widetilde{\Gamma}_k^{\Lambda (n)}(q_1, \ldots, q_n) = \widehat{\delta}(q_1 + \cdots q_n) \int_{x_1 \ldots x_n} \exp(\mathrm{i}(q_1 x_1 + \cdots q_{n-1} x_{n-1})) \Gamma_k^{\Lambda (n)}(x_1, \ldots, x_{n-1}, 0), \tag{47}$$

where $\widehat{\delta}(q) \triangleq (2\pi)^{\mathrm{d}}\delta^{\mathrm{d}}(q)$. In addition, in two-point functions, we solve $q_1 = -q_2 = q$ and recognize that they are functions only of q^2 and we write them $\widetilde{\Gamma}_k^{\Lambda(2)}(q^2)$. In the same vein we denote the Fourier transform of the full propagator at scale 'k' $\widetilde{W}_k^{\Lambda(2)}(q^2) \equiv 1\big/\big(\widetilde{\Gamma}_k^{\Lambda(2)}(q^2)+P_0^k(q^2)\big)$. For a uniform background field Φ equation (24) can thus be rewritten as

$$\partial_k U_k^{\Lambda}[\Phi] = \frac{1}{2}\int_q \frac{\partial_k P_0^k(q^2)}{\widetilde{\Gamma}_k^{\Lambda(2)}[\Phi; q^2] + P_0^k(q^2)}, \tag{48}$$

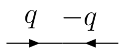
where we have introduced the potential $U_k^{\Lambda} = \Gamma_k^{\Lambda}\big/V$, where V is the volume.

To paraphrase Delamotte [15] this beautiful equation is exact and thus horribly complicated. Mathematically it is a functional parabolic partial derivative equation since both $\Gamma_k^{\Lambda}[\Phi]$ and $\widetilde{\Gamma}_k^{\Lambda(2)}[\Phi; q^2]$ are functionals of Φ. As for canonical theories [14] one can, by functional derivation with respect to the field, deduce from equation (48) an infinite hierarchy of equations for the effective vertices $\widetilde{\Gamma}_k^{\Lambda(n)}(q_1, \ldots, q_n)$. These equations are better represented graphically with the help of Feynman diagrams. The latter will be built from the vertices



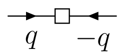
$$= \widetilde{\Gamma}_k^{\Lambda(n)}(q_1, \ldots, q_n) \quad (n \geqslant 2), \tag{49a}$$

the propagator



$$= \widetilde{W}_k^{\Lambda(2)}(q^2), \tag{49b}$$

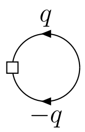
and the insertion



$$= \partial_k \widetilde{P}_0^k(q^2). \tag{49c}$$

For instance (48) takes the form

$$\partial_k U_k^{\Lambda} = \frac{1}{2}$$



$$. \tag{50}$$

Since

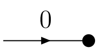
$$(2\pi)^{\mathrm{d}}\frac{\delta\widetilde{\Gamma}_k^{\Lambda(n)}(q_1, \ldots, q_n)}{\delta\widetilde{\Phi}_{-q}} = \widetilde{\Gamma}_k^{\Lambda(n+1)}(q_1, \ldots, q_n, q), \tag{51a}$$

applying the functional $\delta/\delta\widetilde{\Phi}_{-q}$ on a vertex with 'n' legs gives rise to a vertex with 'n+1' legs while, on a propagator, this operation creates a vertex with three legs since

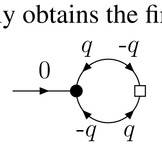
$$(2\pi)^{\mathrm{d}}\frac{\delta\widetilde{W}_k^{\Lambda(2)}(q_1, q_2)}{\delta\widetilde{\Phi}_{-q}} = -\widetilde{W}_k^{\Lambda(2)}(q_1, -r_1)\widetilde{\Gamma}_k^{\Lambda(3)}(r_1, r_2, q_2)\widetilde{W}_k^{\Lambda(2)}(-r_2, -q_2). \tag{51b}$$

With these rules in mind one easily obtains the first equations of the hierarchy

$$\partial_k\widetilde{\Gamma}_k^{\Lambda,(1)}(0) = \partial_k$$

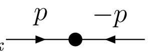


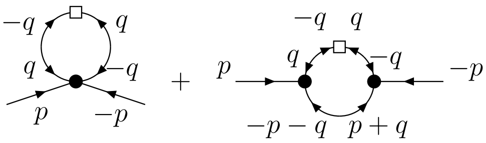
$$= \frac{1}{2}$$



$$= -\frac{1}{2}\int_q \widetilde{\Gamma}_k^{\Lambda(3)}(0, q, -q)\widetilde{W}_k^{\Lambda(2)}(q^2)\partial_k\widetilde{P}_0^k(q^2), \tag{52}$$

and

$$\partial_k \widetilde{\Gamma}_k^{\Lambda\,(2)}(p^2) = \partial_k$$ 

$$= -\frac{1}{2}$$ 

$$
\begin{aligned}
&= -\frac{1}{2}\int_q \widetilde{\Gamma}_k^{\Lambda(4)}(p,q,-q,-p)\widetilde{W}_k^{\Lambda(2)}(q)^2\partial_k \widetilde{P_0^k}(q^2)\\
&\quad + \int_q \widetilde{\Gamma}_k^{\Lambda(3)}(p,q,-p-q)\widetilde{\Gamma}_k^{\Lambda(3)}(-q,-p,p+q)\\
&\quad \times \widetilde{W}_k^{\Lambda(2)}(q)^2\widetilde{W}_k^{\Lambda(2)}(p+q)\partial_k \widetilde{P_0^k}(q^2),
\end{aligned}
\tag{53}
$$

and so on. These tower of equations has exactly the same structure for the canonical and non-canonical theories with the replacement $R_k^\Lambda \to P_0^k$. Flow equations for $\widetilde{\Gamma}_k^{\Lambda(n)}$ of higher orders are obtained in the same vein by making use *ad libitum* of the diagrammatic rules which are deduced from equations (51). Some comments are in order.

- The equation for $\partial_k\widetilde{\Gamma}_k^{\Lambda(n)}$ involves *inter alias* the proper vertex $\partial_k\widetilde{\Gamma}_k^{\Lambda(n+1)}$ and $\partial_k\widetilde{\Gamma}_k^{\Lambda(n+2)}$, therefore the hierarchy never closes. Possible approximations consist in enforcing a closure at some order n [14, 15, 19].
- A little thought reveals that the one-loop structure is present at each order n of the hierarchy and therefore only one integral on internal variables survives.
- All the expressions for the odd $\partial_k\widetilde{\Gamma}_k^{\Lambda(2n+1)}$ include diagrams with at least one odd vertex $\widetilde{\Gamma}_k^{\Lambda(2m+1)}$ ($m \leqslant n$). Therefore, if at some scale k all the odd $\widetilde{\Gamma}_k^{\Lambda(2n+1)}$ happen to vanish they will remain exactly zero at smaller scales k.

5.2. *Sharp cut-off*

To extract the limit $\epsilon \to 0$ of the flow equations one makes use of the 'little lemma' of Morris [13] which states that, for $\epsilon \to 0$

$$\delta_\epsilon(q,k)f(\Theta_\epsilon(q,k),k) \to \delta(q-k)\int_0^1 \mathrm{d}t\, f(t,q), \tag{54}$$

provided that the function $f(\Theta_\epsilon(q,k),k)$ is continuous at $k=q$ in the limit $\epsilon \to 0$, which is the case here. Applying lemma (54) to equation (48) one obtains the flow of the potential

$$\partial_k U_k^\Lambda[\Phi] = \frac{1}{2}k^{d-1}\frac{S_d}{(2\pi)^d}\ln\left(1+\frac{P_0(q^2)}{\Gamma_k^{\Lambda(2)}[\Phi;q^2]}\right), \tag{55}$$

where $S_d = 2\pi^{d/2}\Gamma(d/2)$ is the area of the d-dimensional sphere of radius '1'. The flow equations for the proper vertices $\widetilde{\Gamma}_k^{\Lambda(n)}$ of order $n \geqslant 1$ can also be obtained in the sharp cut-off limit from those of section 5.1 by applying the 'little lemma'. One finds that these equations are identical to those obtained for the first time nearly 25 years ago by Parola and Reatto in the context of the theory of liquids [16, 17]. Equation (55) is still older and was obtained in the early ages of the RG [4, 6, 7].

6. Conclusion

The main result of this paper is contained in equation (14) which states that in a non-canonical, KSSHE-like field theory the Wilsonian action S_k^Λ of the renormalization group coincides with the Helmholtz free energy of the k-system. The average effective action Γ_k^Λ can thus be obtained as a Legendre transform of S_k^Λ (up to a trivial quadratic form). We have derived the RG flow equations for S_k^Λ and Γ_k^Λ and proved some important properties such as parametrization invariance. The exact mapping of section 4 which relates the non-canonical and canonical theories shows interesting features and can also be seen as a practical method to build a KSSHE-like theory from a standard one.

As an illustration let us consider the theory of liquids. Let the fluid be made of identical hard spheres (HS) of diameter σ with additional isotropic pair interactions $v(r_{ij})$ ($r_{ij} = |x_i - x_j|$, x_i is the position of particle 'i'). Since $v(r)$ is an arbitrary function of r in the core, i.e. for $r \leqslant \sigma$, one can assume that $v(r)$ has been regularized in the core in such a way that its Fourier transform $\widetilde{v}_q$ is a well-behaved function of q and that $v(0)$ is a finite quantity. We denote by Ω the domain occupied by the molecules of the fluid. For convenience Ω is supposed to be a cube of side L and periodic boundary (PB) conditions are imposed so that the volume of Ω is $V = L^d$. The fluid is at equilibrium in the grand canonical (GC) ensemble, $\beta = 1/k_\mathrm{B}T$ is the inverse temperature (k_B is Boltzmann's constant) and μ the chemical potential. In addition the particles are subject to an external potential $\psi(x)$ and we will denote by $\nu(x) = \beta(\mu - \psi(x))$ the dimensionless local chemical potential. We stick to notations usually adopted in standard textbooks devoted to the theory of liquids (see, e.g., [37]) and thus denote by $w_0(r) = -\beta v(r)$ *minus* the dimensionless pair interaction. Moreover we restrict ourselves to the case of attractive interactions, i.e. such that $\widetilde{w}_0(q) > 0$ for all q.

In a given GC configuration $\mathcal{C} \equiv (N; x_1 \cdots x_N)$ of the grand canonical ensemble the microscopic density of particles at point x reads $\widehat{\rho}(x|\mathcal{C}) = \sum_{i=1}^N \delta^d(x - x_i)$ and the grand canonical partition function (GCPF) $\Xi[\nu]$ which encodes all the physics of the model at equilibrium is defined as [37]

$$\begin{aligned}
&\Xi[\nu] = \mathrm{Tr}[\exp(-\beta\mathcal{H}_{\mathrm{GC}})],\\
&-\beta\mathcal{H}_{\mathrm{GC}} = -\beta V_{\mathrm{HS}}[\mathcal{C}] + \frac{1}{2}\widehat{\rho}\cdot w_0 \cdot \widehat{\rho} + \overline{\nu}\cdot\widehat{\rho},\\
&\mathrm{Tr}\,[\cdots] = \sum_{N=0}^{\infty} \frac{1}{N!}\int_\Omega \mathrm{d}1\ldots\mathrm{d}n\ldots,
\end{aligned} \tag{56}$$

where $i \equiv x_i$ and $\mathrm{d}i \equiv \mathrm{d}^d x_i$. In equation (56) $\beta V_{\mathrm{HS}}[\mathcal{C}]$ denotes the HS contribution to the configurational energy (i.e. $+\infty$ if there is an overlap of spheres, 0 otherwise) and $\overline{\nu} = \nu + \nu_S$ where $\nu_S = -w_0(0)/2$ is β times the self-energy of a particle. For a given volume V and a given inverse temperature β, $\Xi[\nu]$ is a log-convex functional of the local chemical potential $\nu(x)$ [36, 38].

We now perform a Hubbard–Stratonovich transform to get the KSSHE representation [27]

$$\Xi[\nu] = \mathcal{N}_{w_0^\Lambda}^{-1}\int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi\cdot {w_0^\Lambda}^{-1}\varphi + \ln \Xi_{\mathrm{HS}}\left[\nu - \frac{1}{2}w_0^\Lambda(0) + \varphi\right]\right), \tag{57}$$

where $\widetilde{w}_0^\Lambda(q) = C(q/\Lambda)\widetilde{w}_0(q)$ and $\mathcal{D}\varphi$ is Wegner's measure (cf equation (A.2) of appendix A). We stress that $C(x)$ is the same UV cut-off function we met in section 2; we have used the fact that for $\Lambda \sim 1/\sigma$, $w_0(r)$ and $w_0^\Lambda(r)$ differ only but inside the core. In equation (57)

$\Xi_{\mathrm{HS}}\left[\nu - w_0^{\Lambda}(0)/2 + \varphi\right]$ denotes of course the GCPF of bare hard spheres subject to the local chemical potential $\nu - w_0^{\Lambda}(0)/2 + \varphi$. Comparing (57) with equation (9) we note the one-to-one correspondence $W_R \longleftrightarrow \ln \Xi_{\mathrm{HS}}$ and $w \longleftrightarrow P$. Pair potentials correspond to propagators and W_R is the grand potential of the HS fluid. Note that massive propagators in field theory correspond to attractive Yukawa pair potentials in liquid theory. The RG construction detailed in the core of the paper can be redone (the slight modification due to the introduction of the self-energy $w_0^{\Lambda}(0)$ in equation (57) does not spoil the result). The k-system can thus be identified with a fluid of hard spheres interacting through the pair potentials

$$\widetilde{w}_k^{\Lambda}(q) = (C(q/\Lambda) - C(q/k))\,\widetilde{w}_0(q). \tag{58}$$

In direct space $w_k^{\Lambda}(r)$ is a short range potential equal to $w_0(r)$ for $1/\Lambda \equiv \sigma < r < 1/k$ and equal to 0 for $r > 1/k$, precisely the kind of potential used in numerical simulations involving boxes of side $L = 1/k$. This supports a real space RG interpretation where, at scale 'k', W_k^{Λ} is the Helmholtz free energy of a 'block' of size $1/k$.

Generalizations to repulsive (including Coulomb interactions for instance) or even not definite pair potentials are possible, a detailed analysis will be given elsewhere. Of course this sketchy discussion of the KSSHE theory for a liquid could also be extended in the same vein and with identical conclusions to many other models of condensed matter physics such as the lattice gas or the Ising model.

Acknowledgments

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Appendix A. Gaussian measures and integrals

In this appendix we give some properties of Gaussian integrals used in the main text. Let us consider a real scalar field $\varphi(x)$ defined in a cube $\mathcal{C}_d$ of side L and volume $V = L^d$. We assume periodic boundary conditions, i.e. we restrict ourselves to fields which can be expressed as a Fourier series,

$$\varphi(x) = \frac{1}{V}\sum_{q\in\Lambda}\widetilde{\varphi}_q\,\mathrm{e}^{\mathrm{i}q\cdot x}, \tag{A.1}$$

where $\Lambda = (2\pi/L)\mathbb{Z}^{\mathrm{d}}$ is the reciprocal cubic lattice ($\mathbb{Z}$ set of integers). The reality of φ implies that, for $q \neq 0$ $\widetilde{\varphi}_{-q} = \widetilde{\varphi}_q^{\star}$, where the star means complex conjugation. Following Wegner [2] we define the normalized functional measure $\mathcal{D}\varphi$ as

$$\mathcal{D}\varphi \equiv \prod_{q\in\Lambda}\frac{\mathrm{d}\widetilde{\varphi}_q}{\sqrt{2\pi V}} \tag{A.2a}$$

$$\mathrm{d}\widetilde{\varphi}_q\,\mathrm{d}\widetilde{\varphi}_{-q} = 2\mathrm{d}\,\mathrm{Re}\,\widetilde{\varphi}_q\,\mathrm{d}\,\mathrm{Im}\,\widetilde{\varphi}_q \qquad \text{for} \quad q \neq 0. \tag{A.2b}$$

Equation (A.2) can be conveniently rewritten as

$$\mathcal{D}\varphi = \frac{\mathrm{d}\varphi_0}{\sqrt{2\pi V}}\prod_{q\in\Lambda^{\star}}\frac{\mathrm{d}\,\mathrm{Re}\,\widetilde{\varphi}_q\,\mathrm{d}\,\mathrm{Im}\,\widetilde{\varphi}_q}{\pi V}, \tag{A.3}$$

where the sum in the rhs runs over only the half Λ^* of all the vectors of the reciprocal lattice Λ (for instance those with $q_x \geqslant 0$). With these definitions one has

$$\mathcal{N}_w \equiv \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi \cdot w^{-1} \cdot \varphi\right),$$
$$= \exp\left(\frac{1}{2}\sum_{q\in\Lambda} \ln \widetilde{w}(q)\right) \xrightarrow{L\to\infty} \exp\left(\frac{V}{2}\int_q \ln \widetilde{w}(q)\right), \tag{A.4}$$

where w is definite and positive.

We define the Gaussian measure $\mathrm{d}\mu_w[\varphi] = \mathcal{N}_w^{-1}\mathcal{D}\varphi$ and the Gaussian average $\langle \mathcal{F}[\varphi]\rangle_w = \int \mathrm{d}\mu_w[\varphi]\mathcal{F}[\varphi]$ and recall the well-known Wick's theorem

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle_w = \begin{cases} 0 & \text{if} \quad n \text{ odd,} \\ \displaystyle\sum_{\text{pairs}} w(x_{i_1}, x_{i_2})\cdots w(x_{i_{n-1}}, x_{i_n}) & \text{if} \quad n \text{ even.} \end{cases} \tag{A.5}$$

From Wick's theorem one deduces the important result

$$\langle \exp(J\cdot\varphi)\rangle_w = \exp\left(\tfrac{1}{2}J\cdot w\cdot J\right), \tag{A.6a}$$
$$\langle \exp(iJ\cdot\varphi)\rangle_w = \exp\left(-\tfrac{1}{2}J\cdot w\cdot J\right), \tag{A.6b}$$

where $J(x)$ is a real scalar field. Another consequence of Wick's theorem (A.5) is the following identity involving n Gaussian measures $\mathrm{d}\mu^i_w[\varphi_i]$, $i = 1,\ldots,n$, which is sometimes referred to as the Bogolioubov theorem

$$\int \mathrm{d}\mu_{w_1+\cdots+w_n}[\varphi]\mathcal{F}[\varphi] = \int \prod_{i=1}^{n} \mathrm{d}\mu_{w_i}[\varphi_i]\mathcal{F}[\varphi_1+\cdots+\varphi_n], \tag{A.7}$$

where $\mathcal{F}[\varphi]$ is some arbitrary functional of the field φ.

The last formal consequence of Wick's theorem that we need mention is

$$\int \mathrm{d}\mu_w[\varphi]\mathcal{F}[\varphi+\varphi_0] = \exp(D)\mathcal{F}[\varphi_0], \tag{A.8a}$$

where the functional Laplacian operator D is defined as

$$D \equiv \frac{1}{2}\int_{x,y} w(x,y)\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}. \tag{A.8b}$$

Appendix B. KSSHE theory

We review some properties of a system described by a non-canonical KSSHE partition function

$$Z_\Lambda[J] = \frac{1}{\mathcal{N}_{P_0^\Lambda}}\int \mathcal{D}\varphi \exp(-\mathcal{H}[J,\varphi]), \tag{B.1a}$$
$$\mathcal{H}[J,\varphi] = \frac{1}{2}\varphi\cdot R_0^\Lambda\cdot\varphi - W_R\left[J+\varphi\right], \tag{B.1b}$$

more details will be found in [27] and [33]. In fact, we have already studied the Green functions of the model in section 4.2 since $Z_\Lambda[J]$ is nothing but the special case $Z^\Lambda_{k=0}[J]$. In particular

$$\Phi_\Lambda[J;1] \equiv W_0^{\Lambda(n=1)}(J;1) = R_0^\Lambda(1,2)\cdot\langle\varphi(2)\rangle$$

and the correlations of higher order are given by equations (35) (with $k = 0$). Moreover we also have for $N \geqslant 2$

$$Z_\Lambda^{(n)}[J; 1, \ldots, n] = Z_\Lambda^{-1} \frac{\delta^n Z_\Lambda}{\delta J(1), \ldots, \delta J(n)} = \left\langle Z_R^{(n)}[J + \varphi; 1, \ldots, n] \right\rangle, \tag{B.2}$$

which is not a very useful result except for the case $n = 1$ which gives us the exact relation

$$\Phi_\Lambda [J; 1] = \langle \Phi_R [J + \varphi; 2] \rangle = R_0^\Lambda(1, 2) \langle \varphi(2) \rangle,$$

from which we can guess the MF equation

$$\varphi_{\mathrm{MF}}(1) = P_0^\Lambda(1, 2) \Phi_R[J + \varphi_{MF}; 2], \tag{B.3}$$

which we derive again now on more solid grounds.

The MF approximation is defined as usual as

$$Z_{\Lambda, MF} = \exp(-\mathcal{H}[J, \varphi_{MF}]), \tag{B.4a}$$

$$\left. \frac{\delta \mathcal{H}}{\delta \varphi_{MF}} \right|_J = 0. \tag{B.4b}$$

Clearly the stationarity condition (B.4*b*) coincides with equation (B.3). A short calculation will show that the MF Gibbs free energy is given by [27]

$$\Gamma_{\Lambda, MF}[\Phi] = \Gamma_R[\Phi] - \tfrac{1}{2} \Phi \cdot P_0^\Lambda \cdot \Phi. \tag{B.5}$$

The 2-point vertex function and its inverse are then easily derived from (B.5)

$$\Gamma_{\Lambda, MF}^{(2)} = \Gamma_R^{(2)} - P_0^\Lambda,$$
$$W_{\Lambda, MF}^{(2)} = \left(1 - W_R^{(2)} \cdot P_0^\Lambda\right)^{-1} \cdot W_R^{(2)}.$$

To see that $\Gamma_{\Lambda, MF}[\Phi]$ is a rigorous upper bound to $\Gamma_\Lambda[\Phi]$ we rewrite

$$Z_\Lambda[J] = \langle \exp W_R[J + \varphi] \rangle_{P_0^\Lambda}, \tag{B.6}$$

where the brackets denote a Gaussian average (see appendix A). Applying Young inequalities (4) yields

$$\begin{aligned} Z_\Lambda[J] &\geqslant \langle \exp((J + \varphi) \cdot \Phi - \Gamma_R[\Phi]) \rangle_{P_0^\Lambda} && \forall J, \forall \Phi, \\ &\geqslant \exp(-\Gamma_R[\Phi] + J \cdot \Phi) \langle \exp(\Phi \cdot \varphi) \rangle_{P_0^\Lambda} && \forall J, \forall \Phi, \\ &\geqslant \exp\left(-\Gamma_R[\Phi] + J \cdot \Phi + \tfrac{1}{2} \Phi \cdot P_0^\Lambda \Phi\right) && \forall J, \forall \Phi. \end{aligned} \tag{B.7}$$

Taking the log and making use of (B.5)

$$W_\Lambda[J] \geqslant -\Gamma_{\Lambda, MF}[\Phi] + J \cdot \Phi \qquad \forall J, \forall \Phi,$$

and therefore, for all Φ

$$\Gamma_{\Lambda, MF}[\Phi] \geqslant \sup_J \{J \cdot \Phi - W_\Lambda[J]\} \equiv \Gamma_\Lambda[\Phi],$$

QED. It is obvious that we can extend these result to the k-systems and therefore one has, for any k

$$\overline{\Gamma}_{k, MF}^\Lambda[\Phi] = \Gamma_R[\Phi] - \tfrac{1}{2} \Phi \cdot P_k^\Lambda \cdot \Phi \geqslant \overline{\Gamma}_k^\Lambda[\Phi], \tag{B.8}$$

from which it will be easy for the reader to deduce that for all $0 \leqslant k \leqslant \Lambda$ one has the rigorous bound

$$\Gamma_k^\Lambda[\Phi] \leqslant \Gamma_{\Lambda, MF}[\Phi] \qquad \forall \Phi. \tag{B.9}$$

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